

Chapter 4 Review - Solutions.

- 1 $f(x) = x!$ The factorial definition requires that x be a whole number therefore:

$$\text{domain} \Rightarrow \{x: x \in \mathbb{W}\}$$

the range cannot be described as a list i.e. $\{1, 1, 2, 6, 12, \dots\}$ you cannot predict the next term, therefore it must be described as a recurrence relation. Note that $6! = 6 \times 5!$ so $N! = N \times (N-1)!$

$$\therefore \text{RANGE} \Rightarrow \{u_N: u_N = N u_{N-1}; u_0 = 1 \text{ and } N \in \mathbb{W}\}$$

2. $f(x) = 2x^2 - 6x + 3$ \leftarrow y intercept $(0, 2)$

$$f'(x) = 4x - 6 = 0$$

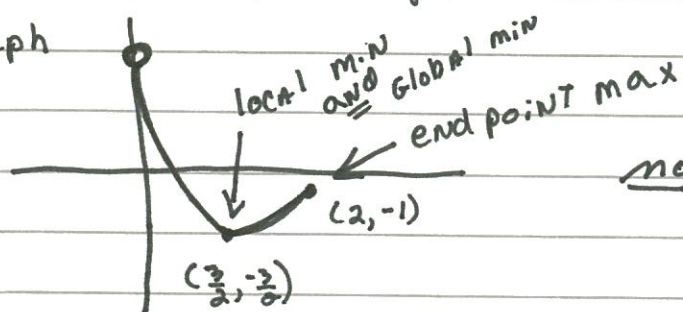
$$x = \frac{3}{2}$$

$$f\left(\frac{3}{2}\right) = 2\left(\frac{3}{2}\right)^2 - 6\left(\frac{3}{2}\right) + 3 = -\frac{3}{2}$$

$$f(2) = 2(2)^2 - 6(2) + 3 = -1$$

- a) critical points: $\left(\frac{3}{2}, -\frac{3}{2}\right)$ and $(2, -1) \leftarrow \text{end point}$
(remember that critical points include end points)

b) graph



endpoint max: $(2, -1)$

- c) global min: $\left(\frac{3}{2}, -\frac{3}{2}\right)$ {succeeded local min title}
no global max.

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3. $f(x) = (x+3)^3 + 1$

$f'(x) = 3(x+3)^2$

$f''(x) = 6(x+3) = 0 \leftarrow \{ \text{to find point of inflection} \}$

$x = -3 \quad f(x) = 1$

point of inflection: $(-3, 1)$

test for concavity

$f''(-4) \quad f''(-3) \quad f''(-2)$
 $\begin{matrix} (-) & & (+) \\ \text{negative} & 0 & \text{positive} \end{matrix}$



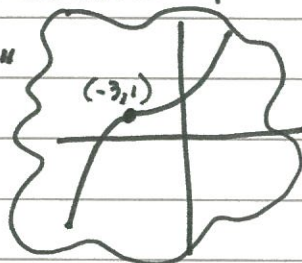
$\cup \leftarrow$ smile is concave up

A "frown" is concave down

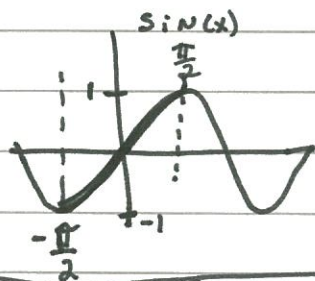
\therefore if $x < -3$ concave down

if $x > -3$ concave up

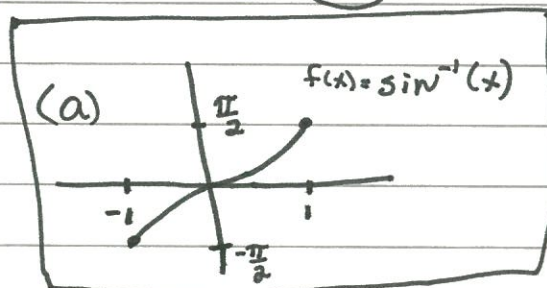
$(-3, 1)$ point of inflection



4.



inverse \rightarrow



domain = $\{x: -1 \leq x \leq 1, x \in \mathbb{R}\}$

range = $\{y: -\frac{\pi}{2} \leq y \leq \frac{\pi}{2}, y \in \mathbb{R}\}$

5(a) $y = \frac{x-2}{x+1}$ {I prefer to graph as I go.}

- find x and y intercepts

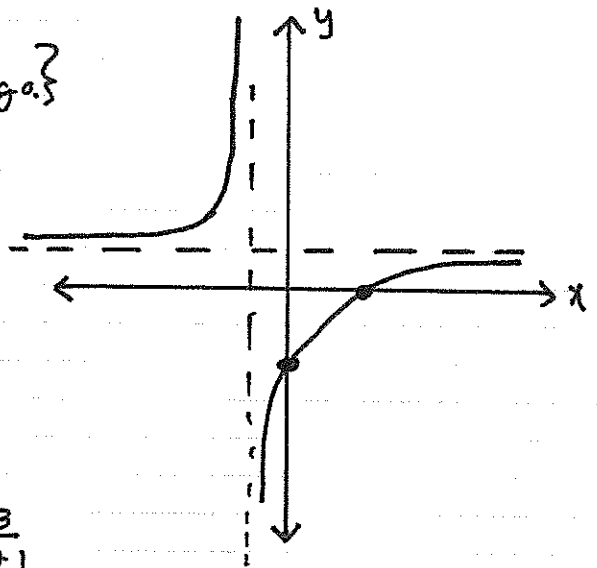
y-int $\rightarrow \frac{0-2}{0+1} = -2$ $(0, -2)$

x-int $\rightarrow x-2=0 \therefore x=2$ $(2, 0)$

- find asymptotes:

vertical asymp. $x = -1$

horizontal asymp. $1 + \frac{-3}{x+1}$
 $y = 1 - \frac{3}{x+1} \therefore y = 1$



- how does graph approach horizontal asymptote: $y = 1$
 as $x \rightarrow \infty^+$ $y \rightarrow 1$ from below {1 - something very small}
 as $x \rightarrow \infty^-$ $y \rightarrow 1$ from above {1 + something very small}

- how does graph approach vertical asymptote: $x = -1$

use $\frac{x-2}{x+1}$ as $x \rightarrow -1$ from left, $y = \frac{\text{neg}}{\text{neg}} = \text{pos } \infty^+$

as $x \rightarrow -1$ from right, $y = \frac{\text{neg}}{\text{pos}} = \text{neg } \infty^-$

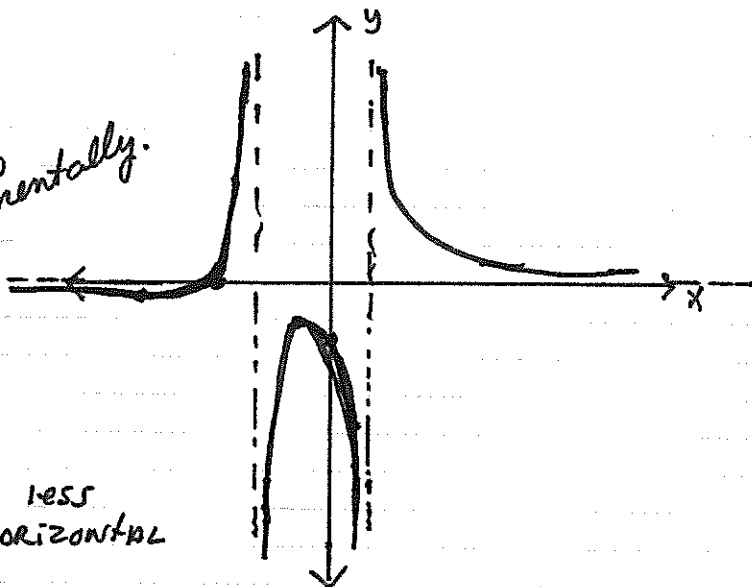
- turning point: $\frac{dy}{dx} = \frac{3}{(x+1)^2} = 0$ no solution \therefore no turning point

- Concavity: $\frac{d^2y}{dx^2} = -\frac{6}{(x+1)^3}$ when $x > -1$ then $\frac{d^2y}{dx^2} < 0$ | concave down \wedge
 when $x < -1$ then $\frac{d^2y}{dx^2} > 0$ | concave up \vee

$$(b) \frac{x+3}{x^2+x-2} = \frac{x+3}{(x+2)(x-1)}$$

- intercepts: y-int: $(0, -\frac{3}{2})$ } *do mentally.*
 x-int: $(-3, 0)$

- vertical asymptotes: $x = -2$
 $x = 1$



- horizontal asymptotes

* when the degree of numerator is less than the denominator then the horizontal asymptote is $y = 0$

- behaviour at asymptotes:

vertical asymptotes: use $\frac{x+3}{(x+2)(x-1)}$

horizontal asymptote:

as $x \rightarrow \infty^+$ $y \rightarrow 0$ from above

as $x \rightarrow \infty^-$ $y \rightarrow 0$ from below

as $x \rightarrow -2$ from left: $\frac{\text{pos}}{\text{neg}} \rightarrow \infty^+$

as $x \rightarrow -2$ from right: $\frac{\text{pos}}{(\text{pos})(\text{neg})} \rightarrow \infty^-$

as $x \rightarrow 1$ from left: $\frac{\text{pos}}{(\text{pos})(\text{neg})} \rightarrow \infty^-$

as $x \rightarrow 1$ from right: $\frac{\text{pos}}{(\text{pos})(\text{pos})} \rightarrow \infty^+$

- turning point:

$$\frac{dy}{dx} = \frac{1(x^2+x-2) - (x+3)(2x+1)}{(x^2+x-2)^2} = \frac{x^2+x-2 - (2x^2+7x+3)}{(x+2)^2(x-1)^2} = \frac{-x^2-6x-5}{(x+2)^2(x-1)^2}$$

$$= -\frac{(x+5)(x+1)}{(x+2)^2(x-1)^2} = 0$$

$$\boxed{x = -5}$$

$$\boxed{y = -\frac{1}{9}}$$

$$\boxed{x = -1}$$

$$\boxed{y = -1}$$

turning points:

$$\boxed{(-5, -\frac{1}{9})} \quad \boxed{(-1, -1)}$$

min \uparrow ~~max~~ * error in text book

$$(c) y = \frac{x^2}{x-3}$$

- intercepts: y-int = (0,0) x-int = (0,0) {mentally}

- asymptotes: vertical $x=3$

as $x \rightarrow 3$ from left: $\frac{pos}{neg} = -\infty$

as $x \rightarrow 3$ from right: $\frac{pos}{pos} = +\infty$

- non-vertical:

$$\begin{array}{r} x+3 + \frac{9}{x-3} \\ x-3 \overline{) x^2 + 0x + 0} \\ \underline{-(x^2 - 3x)} \\ 3x + 0 \\ \underline{-(3x - 9)} \\ 9 \end{array}$$

oblique asymptote $y = x+3$

as $x \rightarrow \infty^+$ $y \rightarrow x+3$ above line

as $x \rightarrow \infty^-$ $y \rightarrow x+3$ below the line

- turning points: use $y = x+3 + \frac{9}{x-3}$

$$\frac{dy}{dx} = 1 - \frac{9}{(x-3)^2} = 0$$

$$1 = \frac{9}{(x-3)^2}$$

$$(x-3)^2 = 9$$

$$(x-3) = \pm 3$$

$$x=0 \text{ or } x=6$$

when $x=0$

$$\cancel{x+3} + \frac{9}{\cancel{x-3}}$$

$$y=0$$

$$(0,0)$$

MAX

when $x=6$

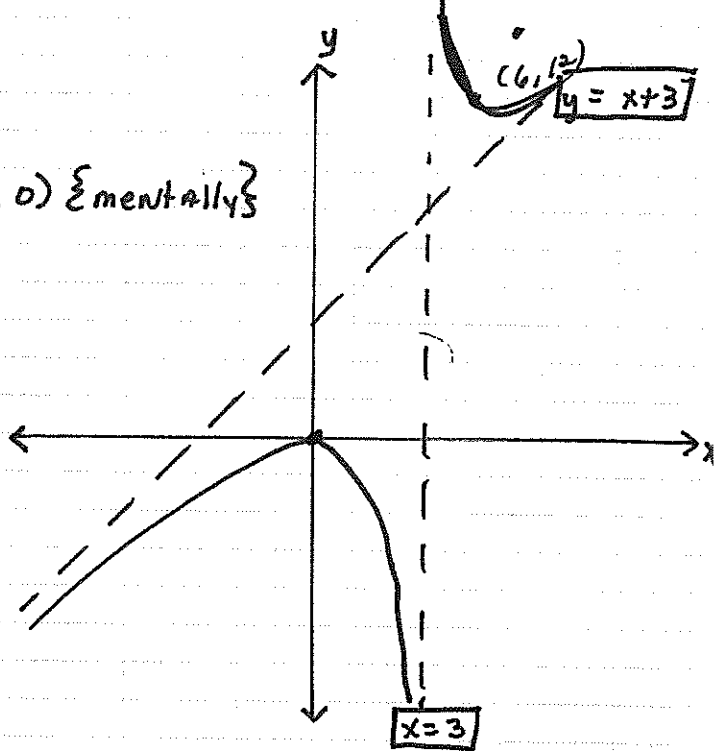
$$\cancel{y=x+3}$$

$$y = \frac{6^2}{3} = 12$$

$$(6,12)$$

MIN

-oops! not enough room for graph!



Nature:

$$\frac{d^2y}{dx^2} = \frac{18}{(x-3)^3}$$

when $x=0$

$\frac{d^2y}{dx^2} < 0$ so (0,0) is a MAX. \cap

when $x=6$

$\frac{d^2y}{dx^2} > 0$ so (6,12) is a MIN. \cup

5(d) $y = \frac{(x-3)(x+6)}{(x-1)(x+2)}$

- y intercept: $\frac{(-3)(6)}{(-1)(2)} = 9$ $(0, 9)$

- x intercepts $(3, 0)$ $(-6, 0)$

- vertical asymptotes $x=1$ $x=-2$

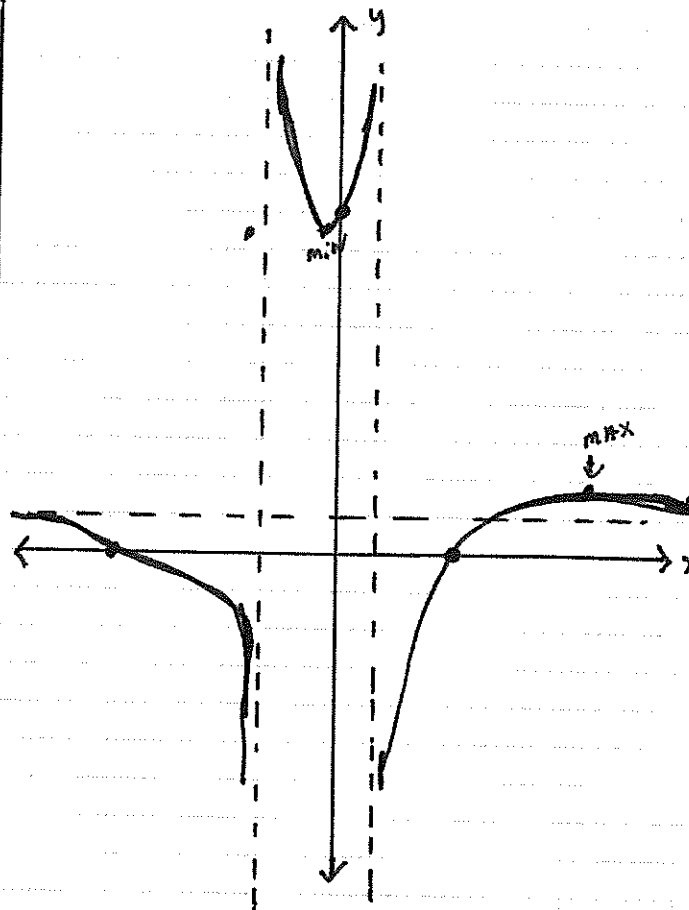
- NON-vert. asymptotes:

$$\frac{(x-3)(x+6)}{(x-1)(x+2)} = \frac{x^2+3x-18}{x^2+x-2}$$

$$x^2+x-2 \sqrt{x^2+3x-18} \quad + \frac{2x-16}{x^2+x-2} = \frac{2(x-8)}{(x+2)(x-1)}$$

$$\frac{-(x^2+x-2)}{2x-16}$$

$y=1$ horiz. asymptote



as $x \rightarrow -2$ from left $\frac{(-ve)}{(neg)(neg)} = \frac{neg}{pos} y < 1$

as $x \rightarrow -2$ from right $\frac{(-ve)}{(pos)(neg)} = \frac{pos}{neg} y > 1$

as $x \rightarrow 1$ from left $\frac{neg}{(pos)(neg)} = \frac{pos}{pos} y > 1$

as $x \rightarrow 1$ from right $\frac{neg}{(pos)(pos)} = \frac{neg}{pos} y < 1$

- TURNING point: use $y = \frac{x^2+3x-18}{x^2+x-2}$

$$\frac{dy}{dx} = \frac{(2x+3)(x^2+x-2) - (x^2+3x-18)(2x+1)}{(x^2+x-2)^2}$$

$$= 2x^3 + 2x^2 + 3x^2 - 4x + 3x - 6 - [2x^3 + x^2 + 6x^2 + 3x - 36x - 18]$$

$$5x^2 - x - 6 - 2x^2 - 6x^2 - 3x + 36x + 18$$

$$-2x^2 + 32x + 12 = 0 \quad \text{use Quadratic Formula}$$

$$x = \frac{-32 \pm \sqrt{32^2 - 4(-2)(12)}}{2(-2)} = \frac{-32 \pm \sqrt{1120}}{-4}$$

$$x = \frac{-32 \pm \sqrt{32^2 - 4(-2)(12)}}{-4}$$

$$x = \frac{-32 \pm \sqrt{1120}}{-4}$$

$x = -0.37$ or $x = 16.37$ $(-0.37, 8.5)$ min

$y = 8.5$ $y = 1.06$ $(16.37, 1.06)$ max

table of signs (use calc values):

| | | | | | |
|----|------|---|----|-------|----|
| -1 | -0.3 | 0 | 10 | 16.37 | 20 |
| - | 0 | + | + | 0 | - |
| ↘ | → | ↗ | ↗ | → | ↘ |
| | min | | | max | |

$$5(e) \quad y = \frac{x^2 + 2x - 3}{x+1} = \frac{(x+3)(x-1)}{(x+1)}$$

$$y\text{-int} \Rightarrow \frac{-3}{1} \quad \boxed{(0, -3)}$$

$$x\text{-int} \rightarrow \boxed{(-3, 0) (1, 0)}$$

$$\text{vert. Asymptote} \quad \boxed{x = -1}$$

As $x \rightarrow -1$ from left: $\frac{(\text{pos})(\text{neg})}{\text{neg}} = \text{pos } \infty^+$

As $x \rightarrow -1$ from right: $\frac{(\text{pos})(\text{neg})}{(\text{pos})} = \text{neg } \infty^-$

Now vert. Asymptote:

$$\begin{array}{r} x+1 \overline{) x^2 + 2x - 3} \\ \underline{-(x^2 + x)} \\ x - 3 \\ \underline{-(x+1)} \\ -4 \end{array}$$

$$y = x+1 - \frac{4}{x+1}$$

$$\text{as } x \rightarrow \infty \quad \boxed{y = x+1} \leftarrow \text{oblique}$$

as $x \rightarrow -\infty$ $y > x+1$ above

as $x \rightarrow +\infty$ $y < x+1$ below

turning points:

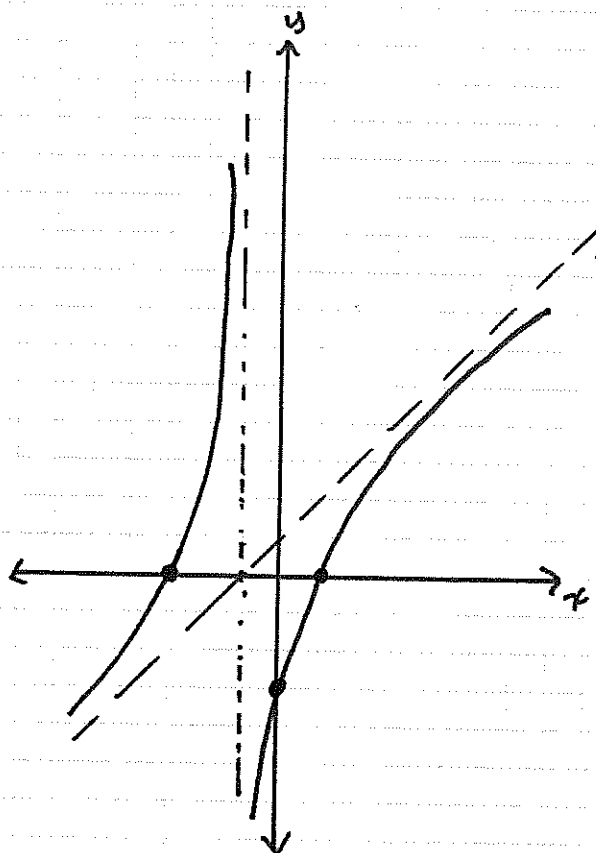
$$\frac{dy}{dx} = 1 + \frac{4}{(x+1)^2} \leftarrow \text{always } > 0$$

no stationary points, always increasing.

inflection points:

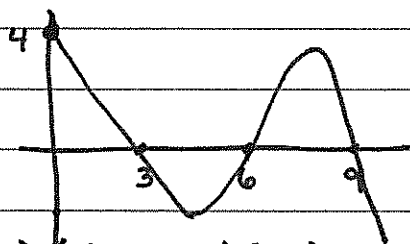
$$\frac{d^2y}{dx^2} = -\frac{8}{(x+1)^3} \leftarrow \text{never } = 0$$

no points of inflection



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6. $y = f(x)$

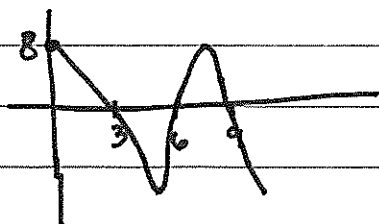


A negative cubic!

points: $(0, 4)$ $(3, 0)$ $(6, 0)$ $(9, 0)$

(a) $y = 2f(x)$ {the y coordinates double}

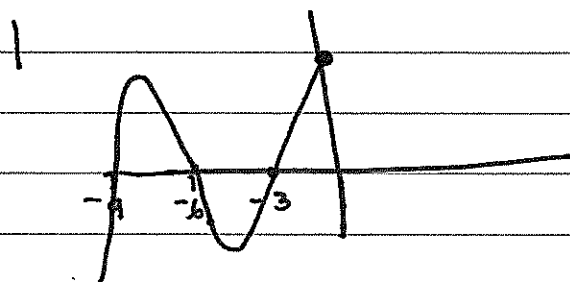
new points $(0, 8)$ $(3, 0)$ $(6, 0)$ $(9, 0)$



the graph is stretched vertically.

(b) $y = f(-x)$ reflect on y-axis

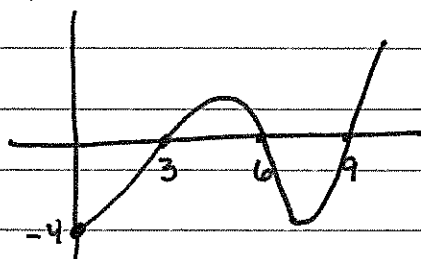
new points $(0, 4)$ $(-3, 0)$ $(-6, 0)$ $(-9, 0)$



{ divide x-coordinates by -1 }

(c) $y = -f(x)$ reflect on x-axis

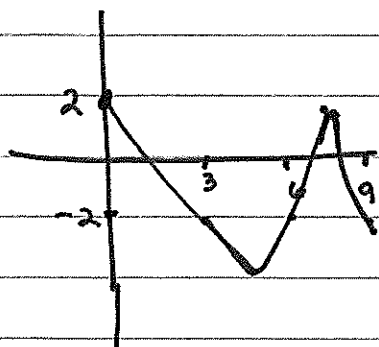
new points $(0, -4)$ $(3, 0)$ $(6, 0)$ $(9, 0)$



{ multiply y coordinates by -1 }

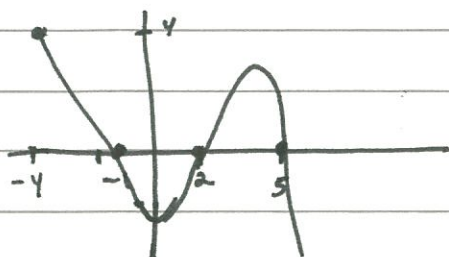
(d) $y = f(x) - 2$ shift graph down 2 units

new points $(0, 2)$ $(3, -2)$ $(6, -2)$ $(9, -2)$



(e) $y = f(x+4)$ graph shifts left 4 units

new points $(-4, 4)$ $(-1, 0)$ $(2, 0)$ $(5, 0)$

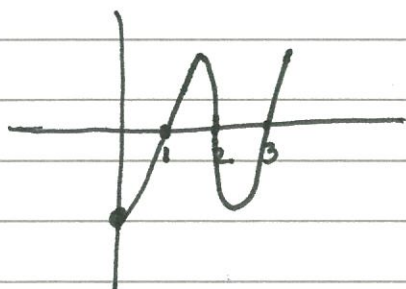


(f) $y = -f(3x)$ divide x by 3

multiply y by -1 (reflect on x axis)

old points: $(0, 4)$ $(3, 0)$ $(6, 0)$ $(9, 0)$

new points: $(0, -4)$ $(1, 0)$ $(2, 0)$ $(3, 0)$



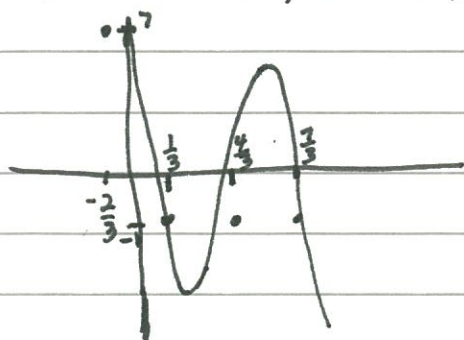
(g) $y = 2f(3x+2) - 1$ $x \rightarrow$ subtract 2 then divide by 3

$y \rightarrow$ multiply by 2 then subtract 1.

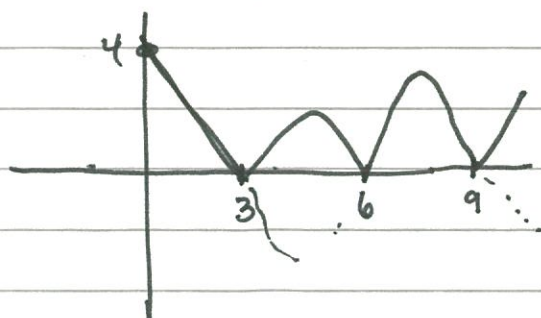
old points: $(0, 4)$ $(3, 0)$ $(6, 0)$ $(9, 0)$

new points: $(-\frac{2}{3}, 7)$ $(\frac{1}{3}, -1)$ $(\frac{4}{3}, -1)$ $(\frac{7}{3}, -1)$ *

book answer is not correct!

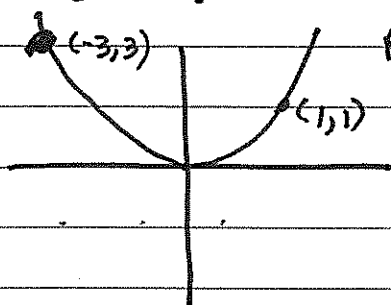


(h) $y = |f(x)| \Rightarrow$ all negative y -values become pos.
points $(0, 4)$ $(3, 0)$ $(6, 0)$ $(9, 0)$ {no change}



all points end up on
or above x -axis

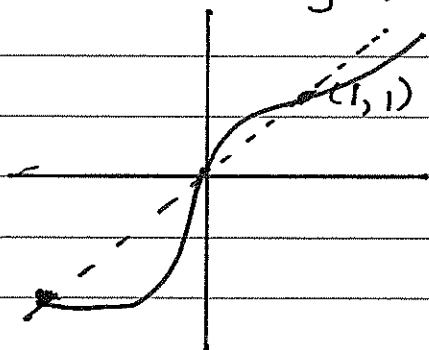
7. (a) $y = |g(x)|$ all negative y -values become positive



easy!

- (b) graph $h(x)$ where $h(x) = g^{-1}(x)$

* reflect on $y=x$



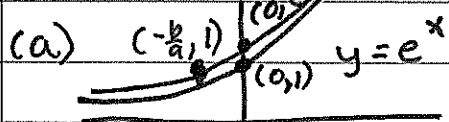
comment:

$$h^{-1}(x) = g(x)$$

easy!

8. sketch e^{ax+b}

subtract b from x then divide by a .
 y stays the same.



$$\therefore \frac{0-b}{a} = -\frac{b}{a} \text{ so } (0, 1) \rightarrow (-\frac{b}{a}, 1)$$

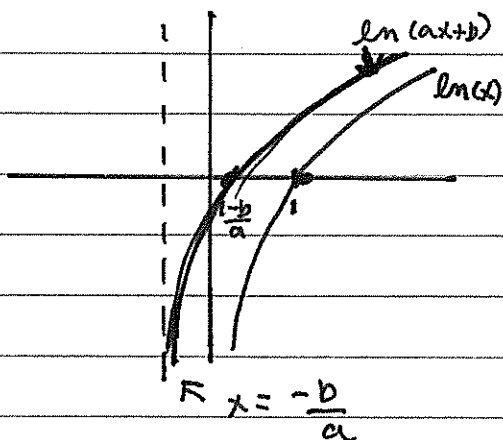
{not as hard as it appears}

* when $x=0$ $y = e^b$ $(0, e^b)$

- (b) sketch $\ln(ax+b)$

- remember $\ln 1 = 0$

- y axis is an asymptote



subtract b from x then divide by a .

$$(1, 0) \rightarrow \frac{1-b}{a}$$

asymptote $x=0 \rightarrow -\frac{b}{a}$