Advanced Higher Formula List

Note: no formulae given in exam – remember everything !

Unit 1

Binomial Theorem

Factorial n

$$n! \overset{def}{=} n \times (n - 1) \times (n - 2) \times \ldots \times 3 \times 2 \times 1$$

Binomial Coefficient

$$^nC_r \equiv \binom{n}{r} \overset{def}{=} \frac{n!}{r!\,(n - r)!}$$

Symmetry Identity

$$\binom{n}{r} = \binom{n}{n - r}$$

Khayyam-Pascal Identity

$$\binom{n}{r - 1} + \binom{n}{r} = \binom{n + 1}{r}$$

Binomial Theorem

$$(x + y)^n = \sum_{r = 0}^{n} {}^nC_r\, x^{n-r}\, y^r = \sum_{r = 0}^{n} {}^nC_r\, x^r\, y^{n-r}$$

General Term in Binomial Theorem

$$^nC_r\, x^r\, y^{n-r}$$

Partial Fractions

Non-Repeated Linear Factor: (ax + b)

$$\frac{S}{ax + b}$$

Repeated Linear Factor: (ax + b) ^{2}

$$\frac{S}{ax + b} + \frac{T}{(ax + b)^2}$$

Irreducible Quadratic Factor: $(ax^2 + bx + c)$

$$\frac{Sx + T}{ax^2 + bx + c}$$

Differential Calculus

n^{th} Derivative

$$\frac{d^n f}{dx^n} \stackrel{def}{=} \underbrace{\frac{d}{dx}\left(\frac{d}{dx}\left(\frac{d}{dx}\cdots\left(\frac{d}{dx}\left(\frac{d}{dx}f\right.\right.\right.\right.}_{n \quad \text{times}}\left.\left.\left.\left.\right)\right)\cdots\right)\right)$$

Product Rule

$$D(f g) = (Df)\, g + f\,(Dg)$$

$$(f g)' = f'g + f g'$$

Quotient Rule

$$D\left(\frac{f}{g}\right) = \frac{(Df)\, g - f\,(Dg)}{g^2}$$

$$(f / g)' = \frac{1}{g^2}(f'g - f g')$$

Reciprocal Trigonometric Functions

$$\sec x \overset{def}{=} \frac{1}{\cos x}$$

$$\operatorname{cosec} x \overset{def}{=} \frac{1}{\sin x}$$

$$\cot x \overset{def}{=} \frac{\cos x}{\sin x}$$

Trigonometric Identities

$$1 + \tan^2 x = \sec^2 x$$

$$1 + \cot^2 x = \operatorname{cosec}^2 x$$

Derivatives of Reciprocal Trigonometric Functions and tan x

$$D(\sec x) = \sec x \tan x$$

$$D(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$$

$$D(\cot x) = -\operatorname{cosec}^2 x$$

$$D(\tan x) = \sec^2 x$$

Derivative of Natural Logarithm

$$D(\ln x) = \frac{1}{x}$$

Derivative of Exponential

$$D(e^x) = e^x$$

$$D(\exp x) = \exp x$$

Applications of Differentiation

Velocity

$$v(t) \overset{def}{=} \frac{ds}{dt} \equiv \dot{s}(t)$$

Acceleration

$$a(t) \overset{def}{=} \frac{dv}{dt} = \frac{d^2 s}{dt^2} \equiv \ddot{s}(t)$$

Integral Calculus

Integrals of e^{x}, x^{-1}, and $\sec^{2} x$

$$\int e^x \, dx = e^x + C$$

$$\int \frac{1}{x} \, dx = \ln |x| + C$$

$$\int \sec^2 x \, dx = \tan x + C$$

Generic Forms of Integration by Substitution

$$\int (Df)\, f \, dx = \frac{1}{2} f^2 + C$$

$$\int \frac{Df}{f} \, dx = \ln |f| + C$$

Area Between a Function and the y-axis

$$A = \int_c^d f^{-1}(y) \, dy$$

Area Between 2 Curves about the y-axis

$$A = \int_c^d (f^{-1}(y) - g^{-1}(y))\ dy$$

Volume of Solid of Revolution about x-axis

$$V = \pi \int_a^b y^2\ dx$$

Volume of Solid of Revolution about y-axis

$$V = \pi \int_c^d x^2\ dy$$

Rectilinear Motion

$$v(t) = \int a(t)\ dt$$

$$s(t) = \int v(t)\ dt$$

Functions and Graphs

Modulus Function

$$|x| \overset{def}{=} \begin{cases} x & (x \geq 0) \\ -x & (x < 0) \end{cases}$$

Modulus of a Function

$$|f| \overset{def}{=} \begin{cases} f & (f \geq 0) \\ -f & (f < 0) \end{cases}$$

Domain and Range of Inverse Trigonometric Functions

$$\text{dom}(\sin^{-1} x) = [-1, 1], \quad \text{ran}(\sin^{-1} x) = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

$$\text{dom}(\cos^{-1} x) = [-1, 1], \quad \text{ran}(\cos^{-1} x) = [0, \pi]$$

$$\text{dom}(\tan^{-1} x) = \mathbb{R}, \quad \text{ran}(\tan^{-1} x) = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

Even and Odd Functions

$$\text{Even}: \quad f(x) = f(-x) \qquad (\forall x \in \text{dom } f)$$

$$\text{Odd}: \quad f(x) = -f(x) \qquad (\forall x \in \text{dom } f)$$

Asymptotes

$$\frac{p(x)}{q(x)} = f(x) + \frac{g(x)}{q(x)}$$

Vertical Asymptote(s) : solve $q(x) = 0$ for x

f constant $\Rightarrow$ Horizontal Asymptote : $y = \text{constant}$

$f(x) = mx + c \Rightarrow$ Oblique Asymptote : $y = mx + c$

Gaussian Elimination

Elementary Row Operations

Interchange 2 or more rows : $R_i \leftrightarrow R_j$

Multiply a row by a non-zero real number : $R_i \to k\, R_j$

Replace a row by adding it to a multiple of another row : $R_i \to R_i + k\, R_j$

No Solutions

$$\left(\begin{array}{ccc|c} a & b & c & j \\ 0 & e & f & k \\ 0 & 0 & 0 & l \end{array}\right) \qquad (l \neq 0)$$

Unique Solution

$$\left(\begin{array}{ccc|c} a & b & c & j \\ 0 & e & f & k \\ 0 & 0 & i & l \end{array}\right) \qquad (i \neq 0)$$

Infinitely Many Solutions

$$\left(\begin{array}{ccc|c} a & b & c & j \\ 0 & e & f & k \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Unit 2

Proof and Elementary Number Theory

Even and Odd Numbers

$$n = 2k$$

$$n = 2k + 1 \quad \text{or} \quad n = 2k - 1$$

Further Differentiation

Derivative of Inverse Function

$$D(f^{-1}) = \frac{1}{(Df) \circ f^{-1}}$$

$$\frac{dx}{dy} = \frac{1}{\left(\frac{dy}{dx}\right)}$$

Derivatives of Inverse Trigonometric Functions

$$D(\sin^{-1} x) = \frac{1}{\sqrt{1 - x^2}}$$

$$D(\cos^{-1} x) = -\frac{1}{\sqrt{1 - x^2}}$$

$$D(\tan^{-1} x) = \frac{1}{1 + x^2}$$

1st Derivative of Parametric Functions

$$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt} = \frac{\dot{y}}{\dot{x}}$$

2nd Derivative of Parametric Functions

$$\frac{d^2y}{dx^2} = \frac{\dot{x}\,\ddot{y} - \dot{y}\,\ddot{x}}{\dot{x}^3} = \frac{d}{dt}\left(\frac{\dot{y}}{\dot{x}}\right) \times \frac{1}{\dot{x}}$$

Applications of Differentiation

Displacement Vector and Distance

$$\mathbf{s}(t) \overset{def}{=} (x(t), y(t)) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j}$$

$$|\mathbf{s}(t)| \overset{def}{=} \sqrt{x^2 + y^2}$$

Velocity Vector, Speed and Direction of Motion (Velocity)

$$\mathbf{v}(t) \overset{def}{=} \frac{d\mathbf{s}}{dt} = (\dot{x}(t), \dot{y}(t)) = \dot{x}(t)\,\mathbf{i} + \dot{y}(t)\,\mathbf{j}$$

$$|\mathbf{v}(t)| \overset{def}{=} \sqrt{\dot{x}^2 + \dot{y}^2}$$

$$\tan\theta = \frac{\dot{y}}{\dot{x}}$$

Acceleration Vector, Magnitude and Direction of Acceleration

$$\mathbf{a}\,(t) \overset{def}{=} \frac{d\mathbf{v}}{dt} = (\ddot{x}\,(t), \ddot{y}\,(t)) = \ddot{x}\,(t)\ \mathbf{i} + \ddot{y}\,(t)\ \mathbf{j}$$

$$\left|\mathbf{a}\,(t)\right| \overset{def}{=} \sqrt{\ddot{x}^2 + \ddot{y}^2}$$

$$\tan\eta = \frac{\ddot{y}}{\ddot{x}}$$

Further Integration

Standard Integrals

$$\int \frac{1}{\sqrt{a^2 - x^2}}\ dx = \int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C$$

$$\int \frac{1}{a^2 + x^2}\ dx = \int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$$

Integration by Parts

$$\int u\,(D\,v) = u\,v - \int (D\,u)\,v$$

$$\int_a^b u\,(D\,v)\ dx = \left[u\ v\right]_a^b - \int_a^b (D\,u)\ v\ dx$$

Separable Differential Equation

$$\frac{dy}{dx} = f\,(x)\,g\,(y)$$

Complex Numbers

Definition of i

$$\mathrm{i}^2 \overset{def}{=} -1$$

Cartesian Form

$$z = x + \mathrm{i}y$$

Complex Conjugate

$$\bar{z} \overset{def}{=} x - \mathrm{i}y$$

Modulus and Principal Argument

$$r \equiv |z| \overset{def}{=} \sqrt{x^2 + y^2}$$

$$\theta \equiv \arg z \overset{def}{=} \tan^{-1}\left(\frac{y}{x}\right) \qquad (\theta \in (-\pi, \pi])$$

Generic Argument

$$\operatorname{Arg} z \overset{def}{=} \{\arg z + 2\pi n : n \in \mathbb{Z}\}$$

Polar Form

$$z = r(\cos\theta + \mathrm{i}\sin\theta) \equiv r \operatorname{cis}\theta$$

Properties of Conjugate, Modulus and Argument

$$|z|^2 = z\bar{z} = x^2 + y^2$$

$$\overline{z \pm w} = \bar{z} \pm \bar{w}$$

$$\overline{zw} = \bar{z}\,\bar{w}$$

$$\overline{\left(\frac{z}{w}\right)} = \frac{\bar{z}}{\bar{w}}$$

$$|zw| = |z|\,|w| \quad , \quad \operatorname{Arg} zw = \operatorname{Arg} z + \operatorname{Arg} w$$

$$\left|\frac{z}{w}\right| = \frac{|z|}{|w|} \quad , \quad \text{Arg}\,\frac{z}{w} = \text{Arg}\,z - \text{Arg}\,w$$

De Moivre's Theorem

$$z = r(\cos\theta + \mathrm{i}\sin\theta) \Rightarrow z^k = r^k(\cos k\theta + \mathrm{i}\sin k\theta)$$

n^{th} Roots of $z^n = w$ with $w = (\cos\theta + \mathrm{i}\sin\theta)$

$$z_k = r^{1/n}\left(\cos\left(\frac{\theta + 2\pi k}{n}\right) + \sin\left(\frac{\theta + 2\pi k}{n}\right)\right)$$

$$(k = 0, 1, 2, \ldots, n-1)$$

Roots of Unity

Solve $z^n = 1$

Sequences and Series

Sum to n Terms of a Sequence

$$S_n \overset{def}{=} \sum_{r=1}^{n} u_r$$

n^{th} Term Given Successive Sums

$$u_n = S_{n+1} - S_n$$

Sum to Infinity

$$S_\infty \overset{def}{=} \lim_{n\to\infty} S_n$$

n^{th} Term of an Arithmetic Sequence

$$u_n = a + (n-1)\,d \qquad (a \in \mathbb{R}, d \in \mathbb{R}\setminus\{0\})$$

Sum to n Terms of an Arithmetic Sequence

$$S_n = \frac{n}{2}\left(2a + (n - 1)d\right)$$

n^{th} Term of a Geometric Sequence

$$u_n = a\,r^{\,n-1} \qquad (a \in \mathbb{R} \setminus\{0\},\, r \in \mathbb{R} \setminus\{0, 1\})$$

Sum to n Terms of a Geometric Sequence

$$S_n = \frac{a\,(1 - r^n)}{1 - r}$$

Sum to Infinity of a Geometric Sequence

$$S_\infty = \frac{a}{1 - r}$$

Expansion of $(1 - x)^{-1}$

$$(1 - x)^{-1} = \frac{1}{1 - x} = 1 + x + x^2 + x^3 + \ldots \overset{def}{=} \sum_{i = 0}^{\infty} x^i$$

Definition of e as a Power Series

$$e \overset{def}{=} \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n = 2 + \frac{1}{2\,!} + \frac{1}{3\,!} + \ldots = \sum_{b = 0}^{\infty} \frac{1}{b\,!}$$

Definition of e^x as a Power Series

$$e^x \overset{def}{=} \lim_{n\to\infty}\left(1 + \frac{x}{n}\right)^n$$

Sum of the Number 1 n times

$$\sum_{r = 1}^{n} 1 = n$$

Sum of the First n Natural Numbers

$$\sum_{r=1}^{n} r = \frac{1}{2} n(n + 1)$$

Sum of the Squares of the First n Natural Numbers

$$\sum_{r=1}^{n} r^2 = \frac{1}{6} n(n + 1)(2n + 1)$$

Sum of the Cubes of the First n Natural Numbers

$$\sum_{r=1}^{n} r^3 = \frac{1}{4} n^2 (n + 1)^2 = \left(\sum_{r=1}^{n} r \right)^2$$

Unit 3

Matrices

Determinant of a 2 x 2 Matrix

$$|A| \equiv \left| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right| = ad - bc$$

Determinant of a 3 x 3 Matrix

$$|A| \equiv \left| \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \right| = a \left| \begin{pmatrix} e & f \\ h & i \end{pmatrix} \right| - b \left| \begin{pmatrix} d & f \\ g & i \end{pmatrix} \right| + c \left| \begin{pmatrix} d & e \\ g & h \end{pmatrix} \right|$$

Inverse of a 2 x 2 Matrix

$$A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Matrix and Determinant Properties

$$A + B = B + A$$

$$(A + B) + C = A + (B + C)$$

$$k(A + B) = kA + kB$$

$$(A + B)^{T} = A^{T} + B^{T}$$

$$(A^{T})^{T} = A$$

$$(kA)^{T} = kA^{T}$$

$$A(BC) = (AB)C$$

$$A(B + C) = AB + AC$$

$$(AB)^{T} = B^{T}A^{T}$$

$$|AB| = |A| \times |B|$$

$$|kA| = k^{n}|A| \quad (k \in \mathbb{R}, n \in \mathbb{N}, A \text{ is } n \times n)$$

$$|A^{T}| = |A|$$

$$|A^{-1}| = \frac{1}{|A|}$$

$$(A^{-1})^{-1} = A$$

$$(A^{-1})^{T} = (A^{T})^{-1}$$

$$(kA^{-1}) = \frac{1}{k}A^{-1}$$

$$(A\,B)^{-1} = B^{-1}A^{-1}$$

Vectors

Vector Product

$$\mathbf{a} \times \mathbf{b} \overset{def}{=} |\mathbf{a}|\ |\mathbf{b}| \sin\theta\ \mathbf{n}$$

$$\mathbf{a} \times \mathbf{b} = (a_2b_3 - a_3b_2)\,\mathbf{i} + (a_3b_1 - a_1b_3)\,\mathbf{j} + (a_1b_2 - a_2b_1)\,\mathbf{k}$$

Properties of the Vector Product

$$\mathbf{i} \times \mathbf{j} = \mathbf{k}$$

$$\mathbf{j} \times \mathbf{k} = \mathbf{i}$$

$$\mathbf{k} \times \mathbf{i} = \mathbf{j}$$

$$\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = \mathbf{0}$$

$$\mathbf{a} \times \mathbf{a} = \mathbf{0}$$

$$\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$$

$$\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) + (\mathbf{a} \times \mathbf{c})$$

$$(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = (\mathbf{a} \times \mathbf{c}) + (\mathbf{b} \times \mathbf{c})$$

Scalar Triple Product

$$[\mathbf{a}, \mathbf{b}, \mathbf{c}] \equiv \mathbf{a} \bullet (\mathbf{b} \times \mathbf{c}) \overset{def}{=} \left|\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}\right|$$

Cartesian Equation of a Plane with normal (a, b, c)^T

$$ax + by + cz = d$$

Vector Equation of a Plane with b, c Parallel to Plane and a in Plane

$$\mathbf{r} = \mathbf{a} + t\,\mathbf{b} + u\,\mathbf{c}$$

Vector Equation of a Line with Direction (a, b, c)^T and a on Line

$$\mathbf{p} = \mathbf{a} + t\,\mathbf{u}$$

Parametric Equations of a Line

$$x = x_1 + at, \quad y = y_1 + bt, \quad z = z_1 + ct$$

Symmetric Equations of a Line

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c} = t$$

Further Sequences and Series

Maclaurin Expansion of a Function

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}\, x^n$$

$$f(x) = f(0) + f'(0)\,x + f''(0)\,\frac{x^2}{2!} + f'''(0)\,\frac{x^3}{3!} + \ldots$$

Specific Maclaurin Expansions

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\, x^{2n+1}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\, x^{2n}$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}$$

$$\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n$$

Binomial Series

$$(1 + x)^r = \sum_{k=0}^{\infty} \binom{r}{k} x^k$$

Further Ordinary Differential Equations

1st Order Integrating Factor Differential Equations

$$\frac{dy}{dx} + P(x)\,y = f(x)$$

solved by multiplying both sides by the Integrating Factor $e^{\int P(x)\,dx}$ and integrating both sides

2nd Order Ordinary Differential Equations

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + c\,y = f(x)$$

solved by adding Complementary Function to Particular Integral.

Further Proof and Number Theory

Linear Diophantine Equations

Solutions of $ax + by = c$ are $x_n = ks + bn$ and $y_n = kt - an$, where $n \in \mathbb{Z}$ and $c = kd$ with $\mathrm{GCD}(a, b) \equiv d = as + bt$.